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Radix-3 Algorithm for Realization of Discrete Fourier Transform

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ABSTRACT
In this paper, a new radix-3 algorithm for realization of discrete Fourier transform (DFT) of length $N = 3^m$ (m = 1, 2, 3,...) is presented. The DFT of length N can be realized from three DFT sequences, each of length $N/3$. If the input signal has length N, direct calculation of DFT requires $O(N^2)$ complex multiplications ($4N^2$ real multiplications) and some additions. This radix-3 algorithm reduces the number of multiplications required for realizing DFT. For example, the number of complex multiplications required for realizing 9-point DFT using the proposed radix-3 algorithm is 60. Thus, saving in time can be achieved in the realization of proposed algorithm.
Keywords: Discrete Fourier transform, fast Fourier transform, radix-3 algorithm

I. INTRODUCTION

Discrete transforms play a significant role in digital signal processing. Among all the discrete transforms, the discrete Fourier transform (DFT) is the most popular transform, and it is mainly due to its usefulness in very large number of applications in different areas of science and technology. The DFT plays a key role in various digital signal processing and image processing applications [1, 2]. Not only it is frequently encountered in many different applications, but also it is computation-intensive. Since DFT is highly computation-intensive, algorithms and architectures are suggested for implementation of the DFT in dedicated very-large-scale integration circuit [3]. DFT and inverse discrete Fourier transform (IDFT) have been regarded as the key technologies for signal processing in orthogonal frequency division multiplexing (OFDM) communication systems. The fast Fourier transform (FFT) is an algorithm that computes the DFT using much less operations than a direct realization of the DFT. The fast realization approach of DFT [4] is known as FFT. FFT algorithms [5, 6] are used for efficient computation of DFT.

In this paper, a new radix-3 algorithm for realization of DFT of length $N = 3^m$ (m = 1, 2, 3,...) is presented. The DFT of length N can be realized from three DFT sequences, each of length $N/3$. If the input signal has length N, direct calculation of DFT requires $O(N^2)$ complex multiplications ($4N^2$ real multiplications) and some additions. This radix-3 algorithm reduces the number of multiplications. For example, the number of complex multiplications required for realizing 9-point DFT using the proposed radix-3 algorithm is 60. Therefore, the hardware complexity and execution time for implementing radix-3 DFT algorithm can be reduced.

The rest of the paper is organized as follows. The proposed radix-3 algorithm for DFT is presented in Section-II. An example for implementation of DFT of length $N = 9$ is presented Section-III. Conclusion is given in Section-IV.

II. PROPOSED RADIX-3 ALGORITHM FOR DFT

The 1-D DFT of input sequence $\{x(n); n = 0, 1, 2,...., N-1\}$ is defined as

$$Y(k) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}} \qquad for\ k = 0, 1, 2, \ldots, N-1 \tag{1}$$

Where $j = \sqrt{-1}$

The $Y(k)$ values represent the transformed data.

Taking $W_N^{kn} = e^{-\frac{j2\pi kn}{N}}$, (1) can be written as

$$Y(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \tag{2}$$

The term W_N^{kn} is known as twiddle factor.

Taking $N=3^m (m=1,2,3,\ldots)$, the N output components $Y(0),Y(1),Y(2),\ldots,Y(N-1)$ are arranged in three groups, namely $Y(3r),Y(3r+1)\ and\ Y(N-3r-1)$, where $r=0,1,2,\ldots,\frac{N}{3}-1$.

The following expressions can be derived from (2).

$$Y(3r)=\sum_{n=0}^{\frac{N}{3}-1}\left[x(n)+x\left(n+\frac{N}{3}\right)+\left(n+\frac{2N}{3}\right)\right]W_N^{kn} \tag{3}$$

Where $k=3r$ and $r=0,1,2,\ldots,\frac{N}{3}-1$.

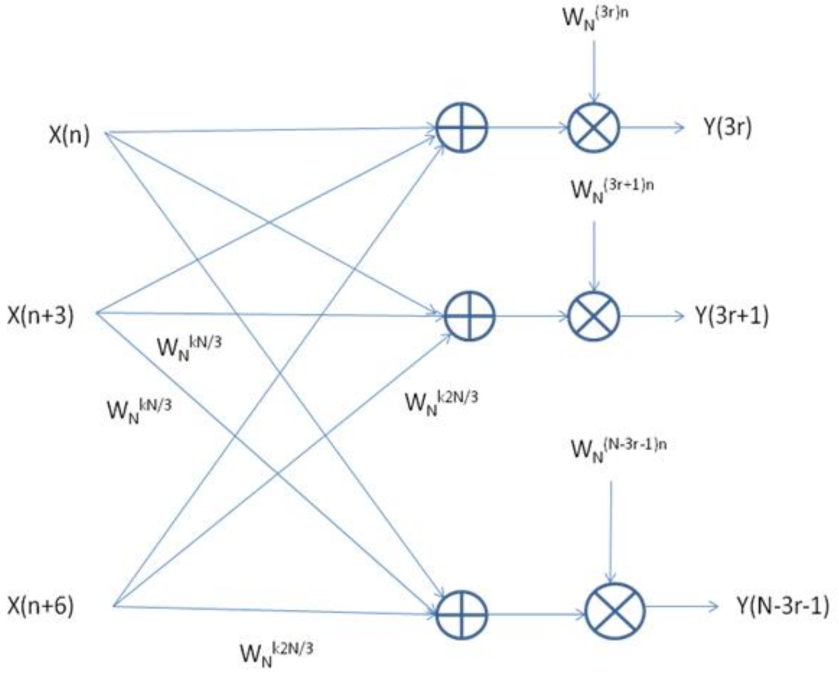
$$Y(3r+1)=\sum_{n=0}^{\frac{N}{3}-1}\left[x(n)+W_N^{k\left(\frac{N}{3}\right)}x\left(n+\frac{N}{3}\right)+W_N^{k\left(\frac{2N}{3}\right)}\left(n+\frac{2N}{3}\right)\right]W_N^{kn} \tag{4}$$

Where $k=3r+1$ and $r=0,1,2,\ldots,\frac{N}{3}-1$.

$$Y(N-3r-1)=\sum_{n=0}^{\frac{N}{3}-1}\left[x(n)+W_N^{k\left(\frac{N}{3}\right)}x\left(n+\frac{N}{3}\right)+W_N^{k\left(\frac{2N}{3}\right)}\left(n+\frac{2N}{3}\right)\right]W_N^{kn} \tag{5}$$

Where $k=(N-3r-1)$ and $r=0,1,2,\ldots,\frac{N}{3}-1$.

The output components $\{Y(k);k=0,1,2,\ldots,N-1\}$ can be realized using (3), (4) and (5) as shown in the butterfly structure of Fig. 1.



$$n=0,1,2,\ldots,\frac{N}{3}-1\ ;\ r=0,1,2,\ldots,\frac{N}{3}-1$$

Fig.1. Butterfly structure for realizing DFT of length *N*

III. EXAMPLE FOR REALIZATION OF DFT OF LENGTH *N* = 9

Substituting *N* = 9 and *k* = 0 in (1), we have

$$Y(0)=\sum_{n=0}^{8}x(n)=\sum_{n=0}^{2}[x(n)+x(n+3)+x(n+6)]$$

$$\Rightarrow Y(0)=\sum_{n=0}^{2}[x(n)+x(n+3)+x(n+6)]\,e^{-\frac{j2\pi kn}{9}} \quad for\ k=0 \tag{6}$$

Substituting *N* = 9 and *k* = 3 in (1), we get

$$Y(3) = \sum_{n=0}^{8} x(n) e^{-\frac{j2\pi\times 3n}{9}}$$

$$= \left[x(0) + x(1) e^{-\frac{j2\pi\times 3}{9}} + x(2) e^{-\frac{j2\pi\times 6}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j2\pi\times 3}{9}} + x(5) e^{-\frac{j2\pi\times 6}{9}} \right] e^{-j2\pi}$$

$$+ \left[x(6) + x(7) e^{-\frac{j2\pi\times 3}{9}} + x(8) e^{-\frac{j2\pi\times 6}{9}} \right] e^{-j4\pi} \quad (7)$$

Since $e^{-j2\pi} = 1$ and $e^{-j4\pi} = 1$, (7) can be expressed as

$$\Rightarrow Y(3) = \sum_{n=0}^{2} [x(n) + x(n+3) + x(n+6)] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 3 \quad (8)$$

Substituting N = 9 and k = 6 in (1), we have

$$Y(6) = \sum_{n=0}^{8} x(n) e^{-\frac{j2\pi\times 6n}{9}}$$

$$= \left[x(0) + x(1) e^{-\frac{j2\pi\times 6}{9}} + x(2) e^{-\frac{j2\pi\times 12}{9}} \right]$$

$$+ \left[x(3) + x(4) e^{-\frac{j2\pi\times 6}{9}} + x(5) e^{-\frac{j2\pi\times 12}{9}} \right] e^{-j4\pi}$$

$$+ \left[x(6) + x(7) e^{-\frac{j2\pi\times 6}{9}} + x(8) e^{-\frac{j2\pi\times 12}{9}} \right] e^{-j8\pi} \quad (9)$$

Since $e^{-j4\pi} = 1$ and $e^{-j8\pi} = 1$, (9) can be written as

$$\Rightarrow Y(6) = \sum_{n=0}^{2} [x(n) + x(n+3) + x(n+6)] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 6 \quad (10)$$

In general (6), (8) and (10) can be expressed as

$$Y(3r) = \sum_{n=0}^{2} [x(n) + x(n+3) + x(n+6)] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 3r\ \ and\ \ r = 0,1\ \&\ 2.$$

Since $W_N^{kn} = e^{-\frac{j2\pi kn}{N}}$, the above expression can be expressed as

$$\Rightarrow Y(3r) = \sum_{n=0}^{2} [x(n) + x(n+3) + x(n+6)] W_9^{(3r)n} \quad for\ k = 3r\ \ and\ \ r = 0,1\ \&\ 2. \quad (11)$$

The above expression is same as (3) for N = 9, $k = 3r$ and $r = 0,1\ \&\ 2$.

Substituting N = 9 and k = 1 in (1), we obtain

$$Y(1) = \sum_{n=0}^{8} x(n) e^{-\frac{j2\pi n}{9}}$$

$$= \left[x(0) + x(1) e^{-\frac{j2\pi}{9}} + x(2) e^{-\frac{j4\pi}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j2\pi}{9}} + x(5) e^{-\frac{j4\pi}{9}} \right] e^{-\frac{j6\pi}{9}}$$

$$+ \left[x(6) + x(7) e^{-\frac{j2\pi}{9}} + x(8) e^{-\frac{j4\pi}{9}} \right] e^{-\frac{j12\pi}{9}}$$

Since $e^{-\frac{j6\pi}{9}} = \frac{-1 - j\sqrt{3}}{2}$ and $e^{-\frac{j12\pi}{9}} = \frac{-1 + j\sqrt{3}}{2}$, the above expression can be written as

$$\Rightarrow Y(1) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 1 \quad (12)$$

Substituting N = 9 and k = 4 in (1), we have

$$Y(4) = \sum_{n=0}^{8} x(n) e^{-\frac{j2\pi\times 4n}{9}}$$

$$= \left[x(0) + x(1) e^{-\frac{j8\pi}{9}} + x(2) e^{-\frac{j16\pi}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j8\pi}{9}} + x(5) e^{-\frac{j16\pi}{9}} \right] e^{-\frac{j24\pi}{9}}$$

$$+ \left[x(6) + x(7) e^{-\frac{j8\pi}{9}} + x(8) e^{-\frac{j16\pi}{9}} \right] e^{-\frac{j48\pi}{9}}$$

Since $e^{-\frac{j24\pi}{9}} = \frac{-1 - j\sqrt{3}}{2}$ and $e^{-\frac{j48\pi}{9}} = \frac{-1 + j\sqrt{3}}{2}$, the above expression can be written as

$$\Rightarrow Y(4) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 4 \qquad (13)$$

Substituting N = 9 and k = 7 in (1), we get

$$Y(7) = \sum_{n=0}^{8} x(n) e^{-\frac{j2\pi \times 7n}{9}}$$

$$= \left[x(0) + x(1) e^{-\frac{j14\pi}{9}} + x(2) e^{-\frac{j28\pi}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j14\pi}{9}} + x(5) e^{-\frac{j28\pi}{9}} \right] e^{-\frac{j42\pi}{9}}$$

$$+ \left[x(6) + x(7) e^{-\frac{j14\pi}{9}} + x(8) e^{-\frac{j28\pi}{9}} \right] e^{-\frac{j84\pi}{9}}$$

Since $e^{-\frac{j42\pi}{9}} = \frac{-1 - j\sqrt{3}}{2}$ and $e^{-\frac{j84\pi}{9}} = \frac{-1 + j\sqrt{3}}{2}$, the above expression can be written as

$$\Rightarrow Y(7) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 7 \qquad (14)$$

In general (12), (13) and (14) can be expressed as

$$Y(3r+1) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}}$$

$$for\ k = 3r+1\ and\ r = 0,1\ \&\ 2.$$

Since $W_N^{kn} = e^{-\frac{j2\pi kn}{N}}$, the above expression can be expressed as

$$\Rightarrow Y(3r+1) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+6) \right] W_9^{(3r+1)n}$$

$$for\ k = 3r+1\ and\ r = 0,1\ \&\ 2. \qquad (15)$$

The above relation is same as (4) for N = 9, $k = 3r+1$ and $r = 0,1\ \&\ 2$.
Substituting N = 9 and k = 2 in (1), we obtain

$$Y(2) = \left[x(0) + x(1) e^{-\frac{j4\pi}{9}} + x(2) e^{-\frac{j8\pi}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j4\pi}{9}} + x(5) e^{-\frac{j8\pi}{9}} \right] e^{-\frac{j12\pi}{9}}$$

$$+ \left[x(6) + x(7) e^{-\frac{j4\pi}{9}} + x(8) e^{-\frac{j8\pi}{9}} \right] e^{-\frac{j24\pi}{9}}$$

$$\Rightarrow Y(2) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 2 \qquad (16)$$

Substituting N = 9 and k = 5 in (1), we get

$$Y(5) = \left[x(0) + x(1) e^{-\frac{j10\pi}{9}} + x(2) e^{-\frac{j20\pi}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j10\pi}{9}} + x(5) e^{-\frac{j20\pi}{9}} \right] e^{-\frac{j30\pi}{9}}$$

$$+ \left[x(6) + x(7) e^{-\frac{j10\pi}{9}} + x(8) e^{-\frac{j20\pi}{9}} \right] e^{-\frac{j60\pi}{9}}$$

$$\Rightarrow Y(5) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 5 \qquad (17)$$

Substituting N = 9 and k = 8 in (1), we have

$$Y(8) = \left[x(0) + x(1) e^{-\frac{j16\pi}{9}} + x(2) e^{-\frac{j32\pi}{9}} \right] + \left[x(3) + x(4) e^{-\frac{j16\pi}{9}} + x(5) e^{-\frac{j32\pi}{9}} \right] e^{-\frac{j48\pi}{9}}$$

$$+ \left[x(6) + x(7) e^{-\frac{j16\pi}{9}} + x(8) e^{-\frac{j32\pi}{9}} \right] e^{-\frac{j96\pi}{9}}$$

$$\Rightarrow Y(8) = \sum_{n=0}^{2} \left[x(n) + \left(\frac{-1 + j\sqrt{3}}{2} \right) x(n+3) + \left(\frac{-1 - j\sqrt{3}}{2} \right) x(n+6) \right] e^{-\frac{j2\pi kn}{9}} \quad for\ k = 8 \qquad (18)$$

In general (16), (17) and (18) can be expressed as

$$Y(9-3r-1)=\sum_{n=0}^{2}\left[x(n)+\left(\frac{-1+j\sqrt{3}}{2}\right)x(n+3)+\left(\frac{-1-j\sqrt{3}}{2}\right)x(n+6)\right]e^{-\frac{j2\pi kn}{9}}$$

$$for\ k=9-3r-1\ \ and\ \ r=0,1\ \&\ 2.$$

$$\Rightarrow Y(9-3r-1)=\sum_{n=0}^{2}\left[x(n)+\left(\frac{-1+j\sqrt{3}}{2}\right)x(n+3)+\left(\frac{-1-j\sqrt{3}}{2}\right)x(n+6)\right]W_9^{(9-3r-1)n}$$

$$for\ k=9-3r-1\ \ and\ \ r=0,1\ \&\ 2. \quad (19)$$

The above relation is same as (5) for $N = 9$, $k = 9-3r-1$ and $r = 0,1\ \&\ 2$.
The output components $Y(0), Y(1), Y(2), ..., Y(8)$ can be realized using (11), (15) and (19) as shown in Fig. 2.

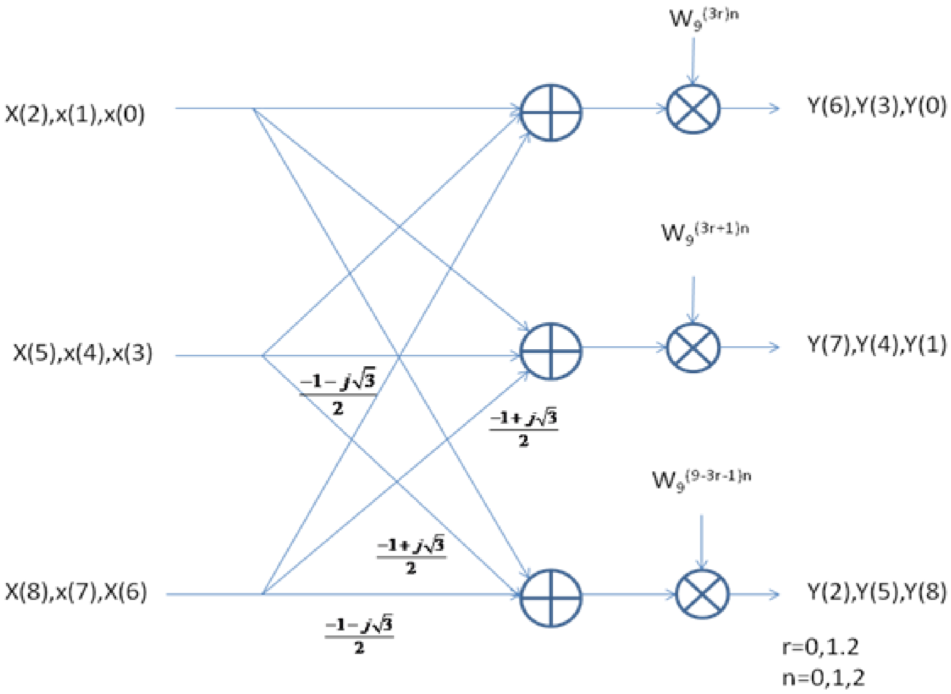


Fig.2. Signal flow graph for realization of DFT of length $N = 9$

IV. CONCLUSION

A new radix-3 algorithm for realization of DFT of length $N = 3^m$ (m = 1, 2, 3,...) has been proposed. The DFT of length N is realized from three DFT sequences, each of length N/3. A butterfly structure for realizing the radix-3 algorithm for DFT of length N is shown and the signal flow graph for DFT of length $N = 9$ is given to clarify the proposal. Direct calculation of DFT of length N requires $O(N^2)$ complex multiplications ($4N^2$ real multiplications) and some additions. This radix-3 algorithm reduces the number of multiplications required for realizing DFT. For example, the number of complex multiplications required for realizing 9-point DFT using the proposed radix-3 algorithm is 60. Therefore, the hardware complexity and execution time for implementing radix-3 DFT algorithm can be reduced.

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